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RESTRAINED WEAK ROMAN DOMINATION IN GRAPHS

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Abstract: Let $G = (V, E)$ be a graph and $f : V \to \{0, 1, 2\}$ be a weak Roman dominating function on G. f is called a restrained weak Roman dominating function, if each vertex $u \in V$ with $f(u) = 0$ is adjacent to another vertex $v \in V$ such that $f(v) = 0$. The weight of a restrained weak Roman dominating function f is defined as $w(f) = f(V) = \sum_{v \in V} f(v)$. The minimum weight of a restrained weak Roman dominating function on G is called the restrained weak Roman domination number of G and is denoted by $\gamma_{rr}(G)$.

Keywords and Phrases: Weak Roman domination, restrained weak Roman domination.

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1. Introduction

Amongst the many areas of research in Mathematics, the ones that can be categorized under graph theory have attracted the interest of many a researcher, owing to the adaptability and the versatility of the subject. A wide variety of topics of research in graph theory have fascinated the minds of serious researchers and one among them is the domination in graphs Ore [13]. Let $G = (V, E)$ be a graph. A *dominating set* D is a subset of V, such that every vertex in V is either in D or is adjacent to some vertex in D. The cardinality of a minimum dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. The set D is said to *dominate* V. The domination in graphs have been extensively studied, a good account of which can be found in Haynes [7]. G. S. Domke et al. [4] defined a variant of the domination in graphs called the restrained domination. For a graph $G = (V, E)$, they defined a *restrained dominating set* as a set $S \subseteq V$ with the property that each vertex in $V\backslash S$ is adjacent to a vertex in S as well as another vertex in $V\backslash S$. The cardinality of the minimum restrained dominating set on G is called the *restrained domination number* of G. A number of results have been produced involving this parameter, a few of which can be found in [2, 5, 6, 21]. Independent of the domination in graphs in conception, but not in its existence, motivated by an article by Ian Stewart [20] on the strategy of the Roman emperor Constantine, the Great, who ruled the Roman empire during the 4th century AD to guard the territories of his empire, Cockayne et al. [3] defined a new parameter called the Roman domination number. For a graph $G = (V, E)$, they defined a *Roman dominating function* (RDF) as a function $f : V \to \{0, 1, 2\}$ such that for any vertex $v \in V$ for which $f(v) = 0$, there exists a vertex $u \in V$ adjacent to v, for which $f(u) = 2$. The weight of a Roman dominating function f defined on $G = (V, E)$, is defined as $w(f) = f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. The Roman dominating function with weight $\gamma_R(G)$ is called the γ_R-*function*. Ever since it was proposed, the Roman domination in graphs has attracted the attention of researchers and several results have been published on this topic, which include [10, 12, 16, 17]. Henning and Hedetniemi [8] defined a new parameter, very similar to the Roman domination number, called the weak Roman domination number, as follows. Let $G = (V, E)$ be a graph and $f : V \to \{0, 1, 2\}$ be a function. A vertex $u \in V$ with $f(u) = 0$ is said to be *undefended* if it is not adjacent to any vertex $v \in V$ such that $f(v) > 0$. The function $f : V \to \{0, 1, 2\}$ is called a *weak Roman dominating function* (WRDF), if for each vertex $u \in V$ for which $f(u) = 0$, there exists a vertex $v \in V$ adjacent to it such that the function

$f' : V \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$, $f'(w) = f(w)$ for all $w \in V \backslash \{u, v\}$, has no undefended vertex. The weight of a weak Roman dominating function f on a graph $G = (V, E)$ is defined as $w(f) = f(V) = \sum_{v \in V} f(v)$. The minimum weight of a weak Roman dominating function defined on a graph G is called the *weak Roman domination number* of G and is denoted by $\gamma_r(G)$. A weak Roman dominating function on G with weight $\gamma_r(G)$ is called a γ_r-*function* on G. Many researchers have studied the parameter weak Roman domination number. Some results produced on this parameter can be found in [11, 14, 15, 18]. Motivated by the definition of restrained domination, P. Roushini Leely Pushpam and S. Padmapriea [19] defined the restrained Roman domination in graphs as follows. Let $G = (V, E)$ be a graph and $f : V \to \{0, 1, 2\}$ be a Roman dominating function on G. f is called a *restrained Roman dominating function*, if each vertex $u \in V$ with $f(u) = 0$ is adjacent to a vertex $v \in V$ with $f(v) = 0$. The weight of a restrained Roman dominating function f is defined as $w(f) = f(V) = \sum_{v \in V} f(v)$. The minimum weight of a restrained Roman dominating function is called the *restrained Roman domination number* of G and is denoted by $\gamma_{rR}(G)$. A restrained Roman dominating function on G with weight $\gamma_{rR}(G)$, is called a γ_{rR}-*function* on G. Contributions to finding the restrained Roman domination number of graphs, are included in [1, 9].

Extending the concept of restrained Roman domination number, we define the restrained weak Roman domination number of a graph as follows. Let $G = (V, E)$ be a graph and $f : V \to \{0, 1, 2\}$ be a weak Roman dominating function on G. f is called a *restrained weak Roman dominating function* (rWRDF), if each vertex $u \in V$ with $f(u) = 0$ is adjacent to a vertex $v \in V$ with $f(v) = 0$. The weight of a restrained weak Roman dominating function f is defined as $w(f) = f(V) = \sum_{v \in V} f(v)$. The minimum weight of a restrained weak Roman dominating function is called the *restrained weak Roman domination number* of G and is denoted by $\gamma_{rr}(G)$. A restrained weak Roman dominating function with weight $\gamma_{rr}(G)$, is called a γ_{rr}-*function*.

In the following, we provide a discussion on the restrained weak Roman domination number of graphs. In the sequel, we only consider graphs $G = (V, E)$ that are simple and connected. Further, we denote the diameter of G by $diam(G)$ and the degree of a vertex $v \in V$ by $deg(v)$.

2. Some Properties of the Restrained weak Roman Domination Number of a Graph

In the discussion that follow, the graph K_1 is considered as the star graph $K_{1,n-1}$ for $n = 1$. It is also considered as the caterpillar graph, with exactly one vertex on the spine and no pendant vertices. A star graph is considered as a caterpillar with exactly one vertex on the spine. We make the following observations.

Observation 2.1. *For any graph G of order n, $\gamma_{rr}(G) = n$ if and only if G is a star.*

Observation 2.2. *There exists no graph G of order n such that $\gamma_{rr}(G) = n - 1$.*

Observation 2.3. *If G contains a cycle, then $\gamma_{rr}(G) \leq n - 2$.*

Observation 2.4. *If G is not the star and has a vertex $v \in V(G)$ such that $deg(v) \geq 2$, then $\gamma_{rr}(G) \leq n - 2$.*

Theorem 2.1. *For any graph G, $\gamma_r(G) \leq \gamma_{rr}(G) \leq \gamma_{rR}(G)$.*
Proof. Every γ_{rr}-function is a WRDF. So, $\gamma_r(G) \leq \gamma_{rr}(G)$. Similarly, every γ_{rR}-function is an rWRDF. Consequently, $\gamma_{rr}(G) \leq \gamma_{rR}(G)$. So, $\gamma_r(G) \leq \gamma_{rr}(G) \leq \gamma_{rR}(G)$.

Theorem 2.2. *If a graph $G = (V, E)$ of order n contains an induced path of length 6, then $\gamma_{rr}(G) < n - 2$.*
Proof. Let the path of length 6 be $v_1, v_2, \ldots, v_7$. Define $f : V \to \{0, 1, 2\}$ by $V_0 = \{v_2, v_3, v_5, v_6\}$, $V_2 = \{v_4\}$ and $V_1 = V \backslash (V_0 \cup V_2)$. Then f defines an rWRDF on G with weight $w(f) = n - 3 < n - 2$. So, $\gamma_{rr}(G) < n - 2$.

Corollary 2.1. *If G is a graph with $diam(G) > 5$, then $\gamma_{rr}(G) < n - 2$.*
Proof. If $diam(G) > 5$, G contains a path of length 6 in G. The result immediately follows from Theorem 2.2.

3. Restrained weak Roman Domination Number of Certain Graphs

In this section, we provide the γ_{rr}-value of certain well known classes of graphs namely, complete graphs, complete bipartite graphs, paths and cycles.

Observation 3.1. *For complete graphs K_n, $\gamma_{rr}(K_n) = \begin{cases} 2, & \text{if } n = 2, \\ 1, & \text{otherwise} \end{cases}$.*

The proof is trivial.

Theorem 3.1. *For the complete bipartite graph $K_{m,n}$, that is not a cycle or a star, $\gamma_{rr}(K_{m,n}) = \begin{cases} 4, & \text{if } m > 3, n > 3, \\ 3, & \text{otherwise.} \end{cases}$*
Proof. Let (X, Y) be the bipartition of the complete bipartite graph $K_{m,n}$, $|X| =$

m, $|Y| = n$, $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$.
Case 1. $m > 3, n > 3$.
Define $f : V \to \{0, 1, 2\}$ by $f(x) = 2$, $f(y) = 2$, for some $x \in X$, $y \in Y$ and $f(z) = 0$ for each $z \in V \backslash \{x, y\}$. This defines an optimal rWRDF on $K_{m,n}$ and $\gamma_{rr}(K_{m,n}) = 4$, $m > 3$, $n > 3$.
Case 2. $m \leq 3, n \geq 3$.
Without loss of generality, let $|m| \leq |n|$. If $m = 2$, $n \geq 3$, the function $f : V \to \{0, 1, 2\}$ defined by $f(w) = \begin{cases} 2, & \text{if } w = x_1 \\ 1, & \text{if } w = y_1 \\ 0, & \text{otherwise} \end{cases}$, is a γ_{rr}-function on $K_{2,n}$, $n \geq 3$. So, $\gamma_{rr}(K_{2,n}) = 3$, $n \geq 3$. If $m = 3$, $n \geq 3$, the function $f : V \to \{0, 1, 2\}$ defined by $f(w) = \begin{cases} 1, & \text{if } w \in \{x_1, x_2, y_1\} \\ 0, & \text{otherwise} \end{cases}$, is a γ_{rr}-function on $K_{3,n}$, $n \geq 3$. So, $\gamma_{rr}(K_{3,n}) = 3$, $n \geq 3$. Hence, if $m \leq 3$, $n \geq 3$, then, $\gamma_{rr}(K_{3,n}) = 3$.

Theorem 3.2. *For paths P_n, $n \geq 1$, $n \neq 3$,*

$$\gamma_{rr}(P_n) = \begin{cases} 2\left\lfloor \frac{n}{4} \right\rfloor + r, & n \equiv r \ (mod\ 4) \ \textit{and}\ r \neq 3 \\ 2\left\lfloor \frac{n}{4} \right\rfloor + 2, & \textit{otherwise} \end{cases}$$

and $\gamma_{rr}(P_3) = 3$.
Proof. For $n \leq 3$, $\gamma_{rr}(P_n) = n$, since $V_0 = V_2 = \phi$ and $V_1 = V(P_n)$. For $n = 4$, $\gamma_{rr}(P_n) = 2$, since $V_0 = \{v_2, v_3\}$, $V_1 = \{v_1, v_4\}$ and $V_2 = \phi$. For $n = 5$, $\gamma_{rr}(P_n) = 3$, since $V_0 = \{v_2, v_3\}$, $V_1 = \{v_1, v_4, v_5\}$ and $V_2 = \phi$, $\gamma_{rr}(P_6) = 4$ with $V_0 = \{v_2, v_3\}$, $V_1 = \{v_1, v_4, v_5, v_6\}$ and $V_2 = \phi$, and when $n = 7$, $\gamma_{rr}(P_n) = 4$ with $V_0 = \{v_2, v_3, v_5, v_6\}$, $V_1 = \{v_1, v_7\}$ and $V_2 = \{v_4\}$. So, for $1 \leq n \leq 7$, $n \neq 3$, we have,

$$\begin{aligned} \gamma_{rr}(P_n) &= \begin{cases} 2\left\lfloor \frac{n}{4} \right\rfloor + r, & n \equiv r \ (\text{mod } 4) \text{ and } r \neq 3 \\ 2\left\lfloor \frac{n}{4} \right\rfloor + 2, & \text{otherwise} \end{cases} \\ \gamma_{rr}(P_3) &= 3. \end{aligned} \tag{1}$$

For the general case, we proceed as follows. Let $P_n : v_1 v_2 \ldots v_n$ be a path on n vertices.
By division algorithm, for any n, we have, $n = 4k + t$, where $0 \leq t < 4$. We shall prove the theorem by induction on $k = \left\lfloor \frac{n}{4} \right\rfloor$, for each of the cases $t = 0, 1, 2, 3$.
Case 1. $t = 0, 1, 2$. If $t = 0$, then $n = 4k$.
Initially, when $k = 2$, we have $n = 8$. The function $f : V \to \{0, 1, 2\}$ defined by,

$f(v_i) = \begin{cases} 1, & \text{for } i \equiv 0, 1 \pmod 4, \\ 0, & \text{otherwise} \end{cases}$, defines a γ_{rr}-function on P_n. Thus, $\gamma_{rr}(P_8) = 4 = 2\left\lfloor \frac{n}{4} \right\rfloor + n \pmod 4$, when $n = 8$. Hence the theorem is true in the initial case. Now assume that the theorem is true for all $k' < k$. That is the theorem is true for all paths of order $4k'$, $k' < k$. We shall prove the theorem for paths P_n, $n = 4k$. Consider the subgraph of H induced by the vertices $v_1, \ldots, v_{4k-4}$. H is a path of order $4(k-1)$. By induction hypothesis, since $k-1 < k$, $\gamma_{rr}(H) = 2\left\lfloor \frac{4(k-1)}{4} \right\rfloor = 2(k-1)$. Let f' be the γ_{rr}-function defined on H. Then, $f : V(P_n) \to \{0, 1, 2\}$ defined by $f(v) = f'(v)$, if $v \in V(H)$, $f(v_{4k-3}) = f(v_{4k}) = 1$, $f(v_{4k-2}) = f(v_{4k-1}) = 0$ defines a γ_{rr}-function on P_n. So, $\gamma_{rr}(P_n) = 2(k-1) + 2 = 2k = 2\left\lfloor \frac{4k}{4} \right\rfloor = \left\lfloor \frac{4k}{4} \right\rfloor + (4k) \pmod 4 = 2\left\lfloor \frac{4k}{4} \right\rfloor = 2\left\lfloor \frac{4k}{4} \right\rfloor + n \pmod 4$. Hence the theorem is true for all paths P_n of order n, $n \pmod 4 = 0$.

The cases $t = 1$ and $t = 2$ can similarly be proved.

Case 2. $t = 3$.
In this case, $n = 4k + 3$. Initially, when $k = 2$, we have $n = 11$. Then, $f : V \to \{0, 1, 2\}$ defined by $f(v_i) = 1$, for $i = 1, 4, 5, 11$, $f(v_i) = 0$ for $i = 2, 3, 6, 7, 9, 10$ and $f(v_8) = 2$ is a γ_{rr}-function on P_{11}. Thus, $\gamma_{rr}(P_{11}) = 6 = 2\lfloor 2 \rfloor + 2 = 2\left\lfloor \frac{11}{4} \right\rfloor + 2 = 2\left\lfloor \frac{n}{4} \right\rfloor + 2$. Hence the theorem is true in the initial case. Now assume that the theorem is true for all paths of order $4k'+3$, where $k' < k$. Let us prove the theorem for P_n, where $n = 4k + 3$. Let H be the subgraph of P_n induced by $v_1, \ldots, v_{n-4}$. By hypothesis, $\gamma_{rr}(H) = 2\left\lfloor \frac{n-4}{4} \right\rfloor + 2$. Let f' be the γ_{rr}-function defined on H. The function $f : V(P_n) \to \{0, 1, 2\}$ defined by $f(v_i) = f'(v_i)$, $1 \leq i \leq n-11$, $f(v_{n-10}) = f(v_{n-7}) = 1$, $f(v_{n-9}) = f(v_{n-8}) = 0$, $f(v_i) = f'(v_{i-4})$, $n - 6 \leq i \leq n$ defines a γ_{rr}-function on P_n. So, $\gamma_{rr}(P_n) = \gamma_{rr}(H) + 2 = 2\left\lfloor \frac{n-4}{4} \right\rfloor + 2 + 2 = 2\left\lfloor \frac{n-4}{4} \right\rfloor + 4 = 2\left\lfloor \frac{n}{4} \right\rfloor + 2$. Hence theorem is true for $n = 4k + 3$.
By induction, the theorem is true, also in this case.
In view of (1), case 1 and case 2, the theorem is proved completely.

Theorem 3.3. *For cycles C_n, $n \geq 4$,*

$$\gamma_{rr}(C_n) = \begin{cases} 2\left\lfloor \frac{n}{4} \right\rfloor + r, & \text{if } n \equiv r \pmod 4 \text{ and } r \neq 3, \\ 2\left\lfloor \frac{n}{4} \right\rfloor + 2, & \text{otherwise} \end{cases}$$

and $\gamma_{rr}(C_3) = 1$.
Proof. We have $\gamma_{rr}(C_3) = \gamma_{rr}(K_3) = 1$.
Let $n > 3$.
Cycles C_n are obtained from paths P_n by joining the pendant vertices of P_n by an edge. Let $v_1, \ldots, v_n$ be a path on n vertices. Then the cycle on these n vertices is obtained by joining v_1 and v_n by an edge. Clearly, the γ_{rr}-labeling of P_n would

define an rWRDF on C_n. So $\gamma_{rr}(C_n) \leq \gamma_{rr}(P_n)$. If possible, let $\gamma_{rr}(C_n) < \gamma_{rr}(P_n)$. Let v be a vertex on C_n with positive weight. If there is a vertex v with weight 1, then v is incident with an edge $e = vw$, such that the weight of w is 1. Removing the edge e from C_n produces a path P of length n. But removing the edge e from C_n would induce an rWRDF function on P with weight $\gamma_{rr}(C_n) < \gamma_{rr}(P_n)$, a contradiction. If there is no vertex with label 1, then we can find a path on v, x, y, z on C_n such that $f(v) = f(z) = 2$, $f(x) = f(y) = 0$. Removing the edge xy from C_n produces a path Q of length n. Defining $g : V(Q) \to \{0, 1, 2\}$ by $g(v) = g(x) = g(y) = g(z) = 1$, $g(w) = f(w)$ for each $w \in V \backslash \{v, x, y, z\}$ defines an rWRDF on Q with weight $\gamma_{rr}(C_n) < \gamma_{rr}(P_n)$, a contradiction. In any case, $\gamma_{rr}(C_n) = \gamma_{rr}(P_n)$.

We note that for any graph G that is not a star, $\gamma_{rr}(G) \leq n-2$. This is evident from Observation 2.1 and Observation 2.2. So, if G is any graph that is not a star then, $\gamma_{rr}(G) \leq n-2$. Hence characterizing those graphs whose γ_{rr}-value is equal to $n-2$ assumes importance.

4. Trees of order n and γ_{rr}-value $n-2$

We shall now characterize those trees whose γ_{rr}-value is two less than their order.

Theorem 4.1. *For any tree T of order n, $\gamma_{rr}(T) = n-2$ if and only if T is a caterpillar with spine length either 2, 3 or 4, $diam(T) > 2$ and all the internal vertices of the spine are of degree 2.*

Proof. In view of Corollary 2.1 to Theorem 2.2, we have to prove the theorem only for trees T with $diam(T) \leq 5$. If $diam(T) = 0$, $T = K_1$ and so $\gamma_{rr}(T) = 1 \neq n-2$. If $diam(T) = 1$, $T = K_2$ and so $\gamma_{rr}(T) = 2 \neq n-2$. If $diam(T) = 2$, T is the star graph $K_{1,n-1}$ and so $\gamma_{rr}(T) = n > n-2$. If $diam(T) = 3$, T is a caterpillar with two vertices on the spine, none of which is an internal vertex and $\gamma_{rr}(T) = n-2$. Now let $diam(T) = 4$. Then the center of T has a single vertex v. There are at least two vertices x, y adjacent to v, each of which is a support vertex. It v has more than two vertices adjacent to it, then the function $f : V \to \{0, 1, 2\}$ defined by $f(v) = f(x) = f(y) = 0$, $f(w) = 1$ if $w \in V \backslash \{v, x, y\}$ defines an rWRDF on T with weight $w(f) = n-3$, consequent to which it follows that $\gamma_{rr}(T) \leq n-3 < n-2$. So $deg(v) = 2$. Then T is a caterpillar with spine length 3, with its only internal vertex having degree 2. If T is such a graph, then, $\gamma_{rr}(T) = n-2$. Now, assume that $diam(T) = 5$. Then the center of T has two vertices u, v. Each of these vertices is adjacent to at least one vertex that is a support vertex. Let u be adjacent to the support x and v be adjacent to the support y. We claim that $deg(u) = deg(v) = 2$. If one of them, say u has $deg(u) > 2$, then u is adjacent to a vertex w that is different from x and v. Define $f : V \to \{0, 1, 2\}$ by $f(u) = f(x) = f(v) = 0$, $f(z) = 1$ if $z \in V \backslash \{u, x, v\}$, defines an rWRDF on T with weight $w(f) = n - 3$. So, $\gamma_{rr}(T) \leq n - 3 < n - 2$. The same is true if $deg(v) > 2$ or $deg(u) > 2$ and $deg(v) > 2$. Then T is a caterpillar with spine length 4, with exactly two internal vertices, both of which have degree 2. If T is such a graph, then $\gamma_{rr}(T) = n - 2$.

5. Cyclic graphs with order n and γ_{rr}-value $n - 2$

In view of Observation 2.1 and Observation 2.2, there are no cyclic graphs of order n with γ_{rr}-value either n or $n-1$. We shall now characterize the cyclic graphs whose γ_{rr}-value is two less than their order. In view of Corollary 2.1, we have to only consider those cyclic graphs with diameter < 6.

Theorem 5.1. *If G contains a cycle of length 5 or 6 with a chord, then $\gamma_{rr}(G) < n - 2$.*

Proof. Let $u_1, \dots, u_5, u_1$ be a cycle of length 5 contained in G with a chord. Without loss of generality, let u_1 and u_3 be adjacent in G. Define $f : V \to \{0, 1, 2\}$ by $V_0 = \{u_2, u_3, u_4\}$, $V_1 = V \backslash V_0$, and $V_2 = \phi$. This defines an rWRDF on G of weight $w(f) = n - 3 < n - 2$. So, $\gamma_{rr}(G) < n - 2$.

On the other hand, let $u_1, \dots, u_6, u_1$ be a cycle of length 6 contained in G with a chord. This chord either transforms the given cycle into one that will contain a triangle and a cycle of length 5 or into one that will contain two cycles of length 4 each.

Case 1. The chord transforms the cycle of length 6 into one that will contain a triangle and a cycle of length 5.

Without loss of generality, let the triangle be on the vertices u_1, u_2 and u_3. Define the function $f : V \to \{0, 1, 2\}$ by $V_1 = \{u_2\}$, $V_2 = \{u_5\}$ and $V_0 = V \backslash (V_1 \cup V_2)$. Then f defines an rWRDF on G of weight $w(f) = n - 3$. So, $\gamma_{rr}(G) \leq n - 3 < n - 2$.

Case 2. The chord transforms the cycle of length 6 into one that will contain two cycles of length 4 each.

Without loss of generality, let the chord join the vertices u_1 and u_4. Define the function $f : V \to \{0, 1, 2\}$ by $V_0 = \{u_3, u_4, u_5\}$, $V_1 = V \backslash V_0$, $V_2 = \phi$. Then f defines an rWRDF on G of weight $w(f) = n - 3$. So, $\gamma_{rr}(G) \leq n - 3 < n - 2$.

Lemma 5.1. *If G is a graph that contains a cycle of length $k \geq 7$ properly, then $\gamma_{rr}(G) < n - 2$.*

Proof. Under the hypothesis, G contains a path of length 6. So by Theorem 2.2, it follows that $\gamma_{rr}(G) < n - 2$.

Lemma 5.2. *If G is a cyclic graph such that $\gamma_{rr}(G) = n - 2$, then G does not contain a cycle of length $k \geq 7$ or a cycle of length 5 or 6 with a chord.*

Proof. The result follows from Theorem 5.1 and Lemma 5.1.

Owing to the small order and size of the graphs involved, the truth of the following Lemmas, Lemma 5.3 to Lemma 5.6 can be easily verified.

Lemma 5.3. *If G is a cyclic graph with $diam(G) = 1$ or 2, then $\gamma_{rr}(G) = n - 2$ if and only if G is one of the following graphs.*

1. *The graph K_3.*
2. *A K_3 with at least one pendant vertex attached to any one of the vertices on it.*
3. *A C_4 or a C_4 with a single chord or a C_4 with a single chord with at least one pendant vertex attached to one end of the chord.*
4. *The graph C_5.*

Proof. There is only one cyclic graph G with $diam(G) = 1$, namely K_3, for which $\gamma_{rr}(G) = n - 2$. Amongst all cyclic graphs with $diam(G) = 2$, only the graphs mentioned in 2, 3 and 4 of the hypothesis satisfy $\gamma_{rr}(G) = n - 2$.

Lemma 5.4. *If G is a cyclic graph with $diam(G) = 3$, then $\gamma_{rr}(G) = n - 2$ if and only if G is one of the following graphs.*

1. *A K_3 with at least one pendant vertex attached to any two vertices on it.*
2. *A C_4 with at least one pendant vertex attached to one of the vertices on it.*
3. *A C_4 with a chord and at least one pendant vertex attached to either ends of the chord.*
4. *The graph C_6.*

Proof. Amongst all the cyclic graphs G with $diam(G) = 3$, only the graphs mentioned in 1, 2, 3 and 4 of the hypothesis satisfy $\gamma_{rr}(G) = n - 2$.

Lemma 5.5. *If G is a cyclic graph with $diam(G) = 4$, then $\gamma_{rr}(G) = n - 2$ if and only if G is the graph obtained from C_4 by attaching at least one pendant vertex to each of two non-adjacent vertices on it.*

Proof. Amongst all the cyclic graphs with $diam(G) = 4$, only the graphs mentioned in the hypothesis of the lemma satisfy $\gamma_{rr}(G) = n - 2$.

Lemma 5.6. *If G is a cyclic graph with $diam(G) = 5$, then $\gamma_{rr}(G) < n - 2$.*

Proof. None of the cyclic graphs G with $diam(G) = 5$ satisfy $\gamma_{rr}(G) = n - 2$ and hence $\gamma_{rr}(G) < n - 2$, by Observation 2.3.

Theorem 5.2. *If G is a cyclic graph, then $\gamma_{rr}(G) = n - 2$ if and only if G is one of the following graphs.*

1. *The graph C_n, $3 \leq n \leq 6$.*
2. *A graph obtained from C_3 by attaching at least one pendant vertex to each of a maximum of two vertices on it.*
3. *A graph obtained from C_4 by attaching at least one pendant vertex to one vertex or to each of two non-adjacent vertices on it.*
4. *A C_4 with a single chord or a graph obtained from a C_4 with a single chord, by attaching at least one pendant vertex to one vertex or to each of two vertices on it*

Proof. The proof follows from Lemma 5.3 to Lemma 5.6.

6. Conclusion

In this paper, we have characterized graphs for which the restrained weak Roman domination number is less than their orders by two. We have also explicitly obtained the restrained weak Roman domination number of certain classes of graph in terms of their orders.

References

[1] Abdollahzadeh Ahangar, H. and Mirmehdipour, S. R., Bounds on the restrained Roman domination number of a graph, Communications in Combinatorics and Optimization, 1(1) (2016), 72-82.

[2] Chen, L., Zeng, W. and Lu, C., NP-completeness and APX-completeness of restrained domination in graphs, Theoretical Computer Science, 448 (2012), 1-8.

[3] Cockayne, E. J., Dreyer, P. A., Hedetniemi, S. M. and Hedetniemi, S. T., Roman domination in graphs, Discrete Mathematics, 278 (2004), 11-22.

[4] Domke, G. S., Restrained Domination in graphs, Discrete Mathematics, 203 (1-3) (1999), 61-69.

[5] Domke, G. S., Hattingh, J. H., Henning, M. A. and Markus, L. R., Restrained domination in trees, Discrete Mathematics, 211(1-3) (2000), 1-9.

[6] Hattingh, J. S. and Joubert, E. J., An upper bound for the restrained domination number of a graph with minimum degree at least two in terms of order and minimum degree, Discrete Applied Mathematics, 157(13) (2009), 2846-2858.

[7] Haynes, T. W., Hedetniemi, S. T. and Slater, P. J., Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[8] Henning, M. A. and Hedetniemi, S. T., Defending the Roman empire - A new Strategy, Discrete Mathematics, 266(1-3) (2003), 239-251.

[9] Ivanovic, M., A Mixed Integer Linear Programming Formulation for Restrained Roman Domination Problem, Theory and Applications of Mathematics & Computer Science, 5(2) (2015), 110-115.

[10] Liu, C. H. and Chang, G. J., Upper bounds on Roman domination number of graphs, Discrete Mathematics, 312(7) (2012), 1386-1391.

[11] Liu, C. S., Peng, S. L. and Tang, C. Y., Weak Roman domination in block graphs, Proceedings of the 27^{th} Workshop on Combinatorial Mathematics and Computation Theory, Providence University, Taichung, Taiwan (2010), 86-89.

[12] Mobaraky, B. P. and Sheikholeslami, S. M., Bounds on Roman domination numbers of graphs, Matematicki Vesnik, 60(4) (2008), 247-253.

[13] Ore, O., Theory of Graphs, Amer. Math. Soc. Colloq. Publ., (American Mathematical Society, Providence, RI), 38 (1962).

[14] Roushini Leely Pushpam, P. and Kamalam, M., Stability of weak Roman domination upon vertex deletion, Asian Journal of Mathematics and Computer Research, 25(2) (2018), 97-105.

[15] Roushini Leely Pushpam, P. and Kamalam, M., Effect of vertex deletion on the weak Roman domination number of a graph, AKCE International Journal of Graphs and Combinatorics, 16(2) (2019), 204-212.

[16] Roushini Leely Pushpam, P. and Malini Mai, T. N. M., On efficiently Roman dominatable graphs, J. Combin. Math. Combin. Comput., 67 (2008), 49-58.

[17] Roushini Leely Pushpam, P. and Malini Mai, T. N. M., Edge Roman domination in graphs, J. Combin Math. Combin Comput. 67 (2009), 175-182.

[18] Roushini Leely Pushpam, P. and Malini Mai, T. N. M., Weak Roman domination in graphs, Discussiones Mathematicae, Graph Theory 31, (2011), 115-128.

[19] Roushini Leely Pushpam, P. and Padmapriea, S., Restrained Roman domination in graphs, Transactions on Combinatorics, 4(1) (2015), 1-17.

[20] Stewart, I., Defend the Roman Empire!, Scientific American, 281(6) (1999), 136-139.

[21] Zverovich, V. and Poghosyan, A., On Roman, Global and Restrained Domination in Graphs, Graphs and Combinatorics, 27(5) (2011), 755-768.